A.M. ≥ G.M. , CBS inequality, etc

1. Obviously, equality holds when $a = b$ ….(1)

Without lost of generality, assume $a > b > 0$.

$$\because \frac{a-b}{2} > 0, \quad a^{\frac{a-b}{2}} < b^{\frac{a-b}{2}} \quad \Rightarrow \quad a^{\frac{a+b}{2}-b} < b^{a-\frac{a+b}{2}} \quad \Rightarrow a^{\frac{a+b}{2}} b^{\frac{a+b}{2}} > a^b b^a \quad \text{….(2)}$$

Since A.M. > G.M. ,
$$\frac{a+b}{2} > a^{\frac{1}{2}} b^{\frac{1}{2}} \quad \Rightarrow \left(\frac{a+b}{2}\right)^{a+b} > a^{\frac{a+b}{2}} b^{\frac{a+b}{2}} \quad \text{….(3)}$$

Combining (1), (2) and (3), $\therefore \quad a^b b^a \le \left(\frac{a+b}{2}\right)^{a+b}$.

2. Obviously the inequality holds when $x = y$.

Without lost of generality, assume $0 < x < y$. Since $m - n > 0$, we have

$$\left(\frac{x}{y}\right) < 1 \Rightarrow \left(\frac{x}{y}\right)^{m-n} < 1 \quad \Rightarrow \left(\frac{x}{y}\right)^m < \left(\frac{x}{y}\right)^n \quad \Rightarrow 1 < 1 + \left(\frac{x}{y}\right)^m < 1 + \left(\frac{x}{y}\right)^n \Rightarrow \left[1 + \left(\frac{x}{y}\right)^m\right]^m < \left[1 + \left(\frac{x}{y}\right)^n\right]^m \quad \text{....(1)}$$

But
$$1 < 1 + \left(\frac{x}{y}\right)^m \text{ and } m > n \quad \Rightarrow \left[1 + \left(\frac{x}{y}\right)^m\right]^n < \left[1 + \left(\frac{x}{y}\right)^m\right]^m \quad \text{….(2)}$$

Combining (1) and (2),
$$\left[1 + \left(\frac{x}{y}\right)^m\right]^n < \left[1 + \left(\frac{x}{y}\right)^n\right]^m \Rightarrow \frac{1}{y^{mn}}\left[x^m + y^m\right]^n < \frac{1}{y^{mn}}\left[x^n + y^n\right]^m$$

As $y^{mn} > 0$, $\therefore (x^m + y^m)^n < (x^n + y^n)^m$.

Alternatively, Let $m = k + n$, w.l.o.g. assume $x \ge y$,

$$(x^m + y^m)^n = (x^n x^k + y^n y^k)^n \le (x^n x^k + y^n x^k)^n = x^{kn} (x^n + y^n)^n = \frac{(x^n)^k}{(x^n + y^n)^k} (x^n + y^n)^{k+n}$$

$$< (x^n + y^n)^{k+n} \quad = \quad (x^n + y^n)^m.$$

3. W.l.o.g., let $a_1 \le a_2 \le \ldots \le a_n$.

(I) If $p, q > 0$, then $a_1^p \le a_2^p \le \ldots \le a_n^p$ and $a_1^q \le a_2^q \le \ldots \le a_n^q$

$\therefore \quad (a_i^p - a_j^p)(a_i^q - a_j^q) \ge 0 \quad \Leftrightarrow \quad a_i^{p+q} + a_j^{p+q} \ge a_i^p a_j^q + a_j^p a_i^q$ ….(1)

Summing the above inequalities from $i = 1, 2, \ldots, n$ (fix j), $\sum_{i=1}^{n} a_i^{p+q} + n a_j^{p+q} \ge a_j^q \sum_{i=1}^{n} a_i^p + a_j^p \sum_{i=1}^{n} a_i^q$

Summing again from $j = 1, 2, \ldots, n$, $n\sum_{i=1}^{n} a_i^{p+q} + n\sum_{j=1}^{n} a_j^{p+q} \ge \sum_{j=1}^{n} a_j^q \sum_{i=1}^{n} a_i^p + \sum_{j=1}^{n} a_j^p \sum_{i=1}^{n} a_i^q$

Since i, j are dummy variables, replace j by i, we have,

$$2n\sum_{i=1}^{n} a_i^{p+q} \ge 2\sum_{i=1}^{n} a_j^p \sum_{i=1}^{n} a_i^q \quad \Leftrightarrow n\sum_{i=1}^{n} a_i^{p+q} \ge \sum_{i=1}^{n} a_j^p \sum_{i=1}^{n} a_i^q \quad \text{….(2)}$$

If $p, q < 0$, then $a_1^p \ge a_2^p \ge \ldots \ge a_n^p$ and $a_1^q \ge a_2^q \ge \ldots \ge a_n^q$.

$\therefore$ The inequality (1) still holds and (2) follows.

(II) If $p < 0$ and $q > 0$, then $a_1^p \ge a_2^p \ge \ldots \ge a_n^p$ and $a_1^q \le a_2^q \le \ldots \le a_n^q$.

If $p > 0$ and $q < 0$, then $a_1^p \le a_2^p \le \ldots \le a_n^p$ and $a_1^q \ge a_2^q \ge \ldots \ge a_n^q$.

In both cases, $(a_i^p - a_j^p)(a_i^q - a_j^q) \le 0$.

As in (I), we have $n\sum_{i=1}^{n} a_i^{p+q} \le \sum_{i=1}^{n} a_j^{p} \sum_{i=1}^{n} a_i^{q}$

4. If $a, b, c, \ldots, k \in \mathbf{Q}^+$. Let m be the L.C.M. of the denominators of $a, b, c, \ldots, k$.

Then $ma, mb, mc, \ldots, mk \in \mathbf{N}^+$. Apply A.M. ≥ G.M., we have,

$$\frac{ma\left(\frac{1}{a}\right)+mb\left(\frac{1}{b}\right)+mc\left(\frac{1}{c}\right)+\ldots+mk\left(\frac{1}{k}\right)}{ma+mb+mc+\ldots+mk} \ge \left[\left(\frac{1}{a}\right)^{ma}\left(\frac{1}{b}\right)^{mb}\left(\frac{1}{c}\right)^{mc}\cdots\left(\frac{1}{k}\right)^{mk}\right]^{\frac{1}{ma+mb+mc+\ldots+mk}}$$

On simplification, we have $\left(\frac{a+b+c+\ldots+k}{n}\right)^{a+b+c+\ldots+k} \le a^a b^b c^c \ldots k^k$ ….(1)

If $a, b, c, \ldots, k \in \mathbf{R}^+$, then $\exists$ infinite sequences $a_i, b_i, c_i, \ldots, k_i \in \mathbf{Q}^+$, $(i = 1, 2, 3, \ldots)$ such that $a_i \to a, b_i \to b, c_i \to c, \ldots, k_i \to k$ as $i \to \infty$.

Now, since $a_i, b_i, c_i, \ldots, k_i \in \mathbf{Q}^+$, $\left(\frac{a_i+b_i+c_i+\ldots+k_i}{n}\right)^{a_i+b_i+c_i+\ldots+k_i} \le a_i^{a_i} b_i^{b_i} c_i^{c_i} \ldots k_i^{k_i}$ ….(2)

Taking $i \to \infty$ on both sides of (2), we have (1) holds for $a, b, c, \ldots, k \in \mathbf{R}^+$.

Equality holds $\Leftrightarrow$ $a = b = c = \ldots = k$.

5. $\prod_{r=1}^{n}(1+x_r) = 1+\sum_{i=1}^{n} x_i + \sum_{i\ne j}^{n} x_i x_j + \sum_{i\ne j\ne k}^{n} x_i x_j x_k + \ldots > 1 + \sum_{r=1}^{n} x_r$

6. $A + B + C = \pi \Rightarrow \tan\frac{C}{2} = \tan\left(\frac{\pi}{2}-\frac{A+B}{2}\right) = \cot\left(\frac{A+B}{2}\right) = \frac{1}{\tan\left(\frac{A}{2}+\frac{B}{2}\right)} = \frac{1-\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}$

$\therefore \quad \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2} = 1$

Put $x = \tan\frac{A}{2}$, $y = \tan\frac{B}{2}$, $z = \tan\frac{C}{2}$, it remains to show: $xy + yz + zx = 1 \Rightarrow x^2 + y^2 + z^2 \ge 1$.

But $(x-y)^2 + (y-z)^2 + (z-x)^2 \ge 0 \Rightarrow x^2 + y^2 + z^2 \ge xy + yz + zx = 1$.

7. **<u>Proof 1</u>**

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}} = \frac{(s-a)(s-b)(s-c)}{abc}, \quad \text{where } s = \frac{a+b+c}{2}$$

$$= \frac{(2s-2a)(2s-2b)(2s-2c)}{8abc} = \frac{(b+c-a)(c+a-b)(a+b-c)}{8abc} \quad \ldots.(1)$$

Since $b + c - a$, $c + a - b$ and $a + b - c$ are positive numbers,

$$\sqrt{(b+c-a)(c+a-b)} \le \frac{(b+c-a)+(c+a-b)}{2} = c \quad \text{(A.M.≥G.M.)} \quad \ldots.(2)$$

Similarly,

$\sqrt{(c+a-b)(a+b-c)} \le a$ ....(3) $\qquad \sqrt{(a+b-c)(b+c-a)} \le b$ ....(4)

(2)×(3)×(4), $(b+c-a)(c+a-b)(a+b-c) \le abc$ ….(5)

Result follows by putting (5) to (1).

Proof 2 $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \left(\frac{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}{3}\right)^3$, (A.M.≥G.M.)

$\le \left(\sin\frac{A/2+B/2+C/2}{3}\right)^3$, since sine function is concave on $[0, \pi]$

$= \left(\sin\frac{A+B+C}{6}\right)^3 = \left(\sin\frac{\pi}{6}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

Proof 3

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)\sin\frac{C}{2} \le \frac{1}{2}\left(1-\sin\frac{C}{2}\right)\sin\frac{C}{2} \le \frac{1}{2}\left[\frac{\left(1-\sin\frac{C}{2}\right)+\sin\frac{C}{2}}{2}\right]^2 = \frac{1}{8}$$

8. **(i)** $\cos A + \cos B + \cos C = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} - \cos(A+B) = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} - \left(2\cos^2\frac{A+B}{2} - 1\right)$

$= 2\cos\frac{A+B}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right) + 1 = 2\cos\frac{A+B}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right) + 1 = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 1$

$\le 4 \times \frac{1}{8} + 1 = \frac{3}{2}$, by No. 7.

(ii) **Proof 1** $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{1}{4}(\sin A + \sin B + \sin C) \le \frac{3}{4}\sin\frac{A+B+C}{3} = \frac{3}{4}\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{8}$

Proof 2 $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \left(\frac{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}{3}\right)^3$, (A.M.≥G.M.)

$\le \left[\cos\frac{1}{3}\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right)\right]^3 = \left[\cos\frac{\pi}{6}\right]^3 = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$, since cosine function is concave on $[0, \pi/2]$

9. **(a)** **(i)** Let $f(x) = \ln x$, then $f'(x) = 1/x > 0$ $\forall x > 0$.

$\therefore$ $f(x)$ is an increasing function.

and $\forall x$, $0 < x \le 1$ $\Leftrightarrow$ $f(x) \le f(1)$ $\Leftrightarrow$ $\ln x \le \ln 1 = 0$ (1)

Now, $\ln a \le \ln b \Rightarrow$ $\ln a - \ln b \le 0$ $\Rightarrow$ $\ln(a/b) \le 0$ $\Rightarrow$ $a/b \le 1$, by (1) $\Rightarrow$ $a \le b$.

(ii) Let $g(x) = x - 1 - \ln x$, $g'(x) = 1 - 1/x$

$\therefore$ $g'(1) = 0$ with $g'(x) \le 0$ $\forall\, 0 < x < 1$ and $g(x) \ge 0$ $\forall x > 1$.

This implies that $g(1) = 0$ is the minimum value of $g(x)$.

$\therefore$ $g(x) = x - 1 - \ln x \ge g(1) = 0$ $\forall x > 0$

$\therefore$ $\ln x \le x - 1$ $\forall x > 0$. The equality holds if and only if $x = 1$.

(b) By the above result, when $t_1, t_2, \ldots, t_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive numbers, we have

$$\alpha_1 \ln t_1 + \alpha_2 \ln t_2 + \ldots + \alpha_n \ln t_n \le \alpha_1 (t_1 - 1) + \alpha_2 (t_2 - 1) + \ldots + \alpha_n (t_n - 1)$$

$$= (\alpha_1 t_1 + \alpha_2 t_2 + \ldots + \alpha_n t_n) - (\alpha_1 + \alpha_2 + \ldots + \alpha_n) = 1 - 1 = 0$$

$\therefore \quad \ln\left(t_1^{\alpha_1} t_2^{\alpha_2} \ldots . t_n^{\alpha_n}\right) \le 0 \qquad \Rightarrow t_1^{\alpha_1} t_2^{\alpha_2} \ldots . t_n^{\alpha_n} \le 1$

Now, let $t_i = \dfrac{(m_1 + m_2 + \ldots + m_n) a_i}{m_1 a_1 + m_2 a_2 + \ldots + m_n a_n}$, $\quad \alpha_i = \dfrac{m_i}{m_1 + m_2 + \ldots + m_n}$ $\quad \forall\, i = 1, 2, \ldots, n.$

It can be seen that $\alpha_1 t_1 + \alpha_2 t_2 + \ldots . + \alpha_n t_n = \alpha_1 + \alpha_2 + \ldots . + \alpha_n = 1$

$\therefore$ By the above, $t_1^{\alpha_1} t_2^{\alpha_2} \ldots . t_n^{\alpha_n} \le 1$

$$\prod_{i=1}^{n} \left(\frac{(m_1 + m_2 + \ldots + m_n) a_i}{m_1 a_1 + m_2 a_2 + \ldots + m_n a_n} \right)^{\frac{m_i}{m_1 + m_2 + \ldots + m_n}} \le 1 \Rightarrow \frac{(m_1 + m_2 + \ldots + m_n)}{m_1 a_1 + m_2 a_2 + \ldots + m_n a_n} \prod_{i=1}^{n} \left(a_i^{m_i}\right)^{\frac{1}{m_1 + m_2 + \ldots + m_n}} \le 1$$

$$\therefore \quad a_1^{m_1} a_2^{m_2} \ldots . a_n^{m_n} \le \left(\frac{m_1 a_1 + m_2 a_2 + \ldots . + m_n a_n}{m_1 + m_2 + \ldots . + m_n} \right)^{m_1 + m_2 + \ldots + m_n}$$

10. **(i)** $\dfrac{x^{p+1} - 1}{p+1} \ge \dfrac{x^p - 1}{p} \Leftrightarrow px^{P+1} - p \ge px^p + x^p - p - 1 \Leftrightarrow px^{P+1} - px^p + x^p + 1 \ge 0$

$\Leftrightarrow (x - 1)\,[\,px^p - x^{p-1} - x^{p-2} - \ldots - x - 1] \ge 0 \Leftrightarrow (x - 1)^2\, [px^{p-1} + (p - 1)x^{p-2} + \ldots + 3x^2 + 2x + 1] \ge 0$

$\Leftrightarrow$ True statement since the expression is the product of a complete square and a positive quantity

Equality holds $\Leftrightarrow x - 1 = 0 \Leftrightarrow x = 0$

(ii) **(a)** By (i), $\dfrac{x_i^m - 1}{m} \ge \dfrac{x_i^{m-1} - 1}{m - 1} \ge \ldots \ge \dfrac{x_i - 1}{1} \quad \Rightarrow \displaystyle\sum_{i=1}^{n} \frac{x_i^m - 1}{m} \ge \sum_{i=1}^{n} \frac{x_i - 1}{1} = \sum_{i=1}^{n} x_i - n = n - n = 0$

$$\Rightarrow \quad \sum_{i=1}^{n} x_i^m - n \ge 0 \Rightarrow \sum_{i=1}^{n} x_i^m \ge n$$

(b) Suppose on the contrary $\exists\, x_j$ s.t. $x_j \ne 1$, then $\dfrac{x_j^m - 1}{m} > \dfrac{x_j - 1}{1}$, since equality holds $\Leftrightarrow$ x = 1.

$\forall\, i \ne j$, we have $\dfrac{x_i^m - 1}{m} \ge \dfrac{x_i - 1}{1}$.

Summing up all inequalities from i = 1, 2, …, n including the j^{th} term, we have

$\displaystyle\sum_{i=1}^{n} \frac{x_i^m - 1}{m} > \sum_{i=1}^{n} \frac{x_i - 1}{1}$. Strict inequality holds since there is a strict inequality in the j^{th} term.

Following the same method in (a), we have $\displaystyle\sum_{i=1}^{n} x_i^m > n$, which contradicts to the given $\displaystyle\sum_{i=1}^{n} x_i^m = n$.

(iii) Let $A = \dfrac{y_1 + y_2 + \ldots + y_n}{n}$, $\quad x_i = \dfrac{y_i}{A}$, i = 1, 2, …, n. Then

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{y_i}{A} = \frac{1}{A} \sum_{i=1}^{n} y_i = \frac{nA}{A} = n \ge n \Rightarrow \sum_{i=1}^{n} x_i^m \ge n, \quad \text{by (ii)(a).}$$

$$\sum_{i=1}^{n} \left(\frac{y_i}{A}\right)^m \ge n \quad \Rightarrow \frac{1}{n} \sum_{i=1}^{n} y_i^m \ge A^m \Rightarrow \frac{y_1^m + y_2^m + \ldots . + y_n^m}{m} \ge \left(\frac{y_1 + y_2 + \ldots . + y_n}{m} \right)^m$$

If the equality holds, by (ii)(b), then either m = 1 or $x_i = 1\ \forall\, i = 1, 2, \ldots, n$.

i.e. either m = 1 or or $y_1 = y_2 = \ldots . = y_n$.